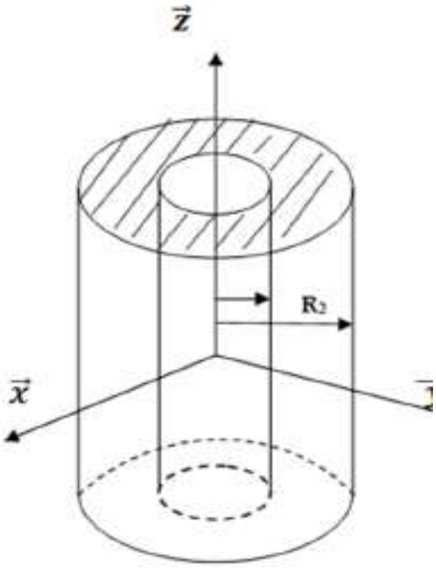
UCA - Marrakech -FP –Safi -Année Universitaire 19/20
Mécanique de solide

Planche 2 : Géométrie des masses

Exercice 1



Matrice d'inertie :

a) Cylindre creux de rayons R_1, R_2 et hauteur H et de masse M

Les trois plans $(o, \vec{x}, \vec{y})$, $(o, \vec{x}, \vec{z})$, $(o, \vec{y}, \vec{z})$ sont des plans de symétrie matérielles

$(\vec{x}, \vec{y}, \vec{z})$ est une base principale d'inertie

$\Longrightarrow$ lamatrice d'inertie est diagonale

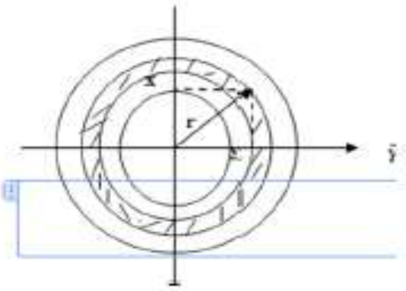
$$[I_{(S)}]_{(o,\vec{x},\vec{y},\vec{z})} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \qquad A = \int (y^2 + z^2$$

$z^2)dm$

$$C = \int (x^2 + y^2)dm$$

On a : $dV = 2\boldsymbol{\pi}\, r\, dr\, dz$

On a : $r^2 = x^2 + y^2$



$\vec{y}$

$$C = \int r^2 . dm \quad ; dm = \rho . dv$$

$$C = \int r^2 . \rho . 2\pi\, r\, dr . dz$$

$$C = 2\pi . \rho \int_{R_1}^{R_2} r^3 . \int_{-\frac{H}{2}}^{\frac{H}{2}} dz$$

$$C = 2\pi . \rho . \frac{1}{4}(R_2^4 - R_1^4)\ . H$$

Avec

$$\rho = \frac{M}{V} = \frac{M}{\pi(R_2^2 - R_1^2) . H}$$

$$C = 2\pi . \frac{M}{\pi(R_2^2 - R_1^2) . H} . \frac{1}{4} . (R_2^4 - R_1^4)\ . H$$

$$C = \frac{1}{2} M . \frac{(R_2^4 - R_1^4)}{R_2^2 - R_1^2} = \frac{1}{2} M . (R_2^2 + R_1^2)$$

$$\boxed{C = \frac{1}{2} M (R_2^2 + R_1^2)}$$

On a : $(o,\vec{x})$ joue le même rôle avec $(o,\vec{y}) \Longrightarrow A = B$

A+B= $\int (y^2 + z^2 + x^2 + z^2) dm = \int (x^2 + y^2) dm + 2\int z^2\, dm$

A+B= $C + 2\int z^2\, dm \quad \Longrightarrow$ A= B = $\frac{1}{2}\, C + \int z^2 dm$

On a : $\int z^2 dm = \rho . \int z^2 dv$

$$= \rho \int z^2 . 2\pi\, r\, dr . dz$$

$$= 2\pi\rho \int_{R_1}^{R_2} r\, dr \int_{-H/2}^{H/2} z^2 dz$$

$$= 2\pi\rho \frac{1}{2}(R_2^2 - R_1^2) . \frac{1}{3}(\frac{H^3}{8} + \frac{H^3}{8})$$

$$= \pi\rho . (R_2^2 - R_1^2)\frac{H^3}{12}$$

$$\int z^2 dm = \pi \frac{M}{\pi (R_2^2 - R_1^2) . H} \quad (R_2^2 - R_1^2)\frac{H^3}{8}$$

$$= \frac{MH^2}{12} \qquad A = B = \frac{1}{4} M (R_2^2 + R_1^2) + \frac{MH^2}{12}$$

$$[I_{(S)}]_o = \begin{bmatrix} \frac{M}{4}(R_2^2 + R_1^2) + \frac{MH^2}{12} & 0 & 0 \\ 0 & \frac{M}{4}(R_2^2 + R_1^2) + \frac{MH^2}{12} & 0 \\ 0 & 0 & \frac{M}{2}(R_2^2 + R_1^2) \end{bmatrix}_{(o,\vec{x},\vec{y},\vec{z})}$$

2ème méthode :

$$C = C_2 - C_1 = \frac{M_2 R_2^2}{2} - \frac{M_4 R_1^2}{2}$$

$M = M_2 - M_1 = \rho\pi(R_2^2 - R_1^2)H \,, M_2 = \rho\pi R_2^2 . H \;\, , M_1 = \rho\pi R_1^2 \mathrm{H}$

$$C = \rho\pi R_2^2 . H \frac{R_2^2}{2} - \rho\pi R_1^2 H \frac{R_1^2}{2}$$

$$= \rho\pi(\frac{R_2^4}{2} . H - \frac{R_1^4}{2})$$

$$on\ a: \quad \rho\pi = \frac{M}{H(R_2^2 - R_1^2)}$$

$$C = \frac{M}{H(R_2^2 - R_1^2)} \ .H\left(\frac{R_2^4}{2} - \frac{R_1^4}{2}\right) = \frac{1}{2}M\frac{R_2^4 - R_1^4}{R_2^2 - R_1^2} = \frac{M}{2}(R_2^2 + R_1^2)$$

$$A = B = A_2 - A_1 = \frac{M_2}{4}R_2^2 + M_2\frac{H^2}{12} - \frac{M_1}{4}R_1^2 - M_1\frac{H^2}{12}$$

$$A = B = A_2 - A_1 = \frac{M_2}{4}R_2^2 - \frac{M_1}{4}R_1^2 + M_2\frac{H^2}{12} - M_1\frac{H^2}{12}$$

$$= \frac{1}{4}[\rho\pi R_2^2.HR_2^2 - \rho\pi R_1^2.HR_1^2] + \frac{1}{12}[\rho\pi R_2^2.H.H^2 - \rho\pi R_1^2H.H^2]$$

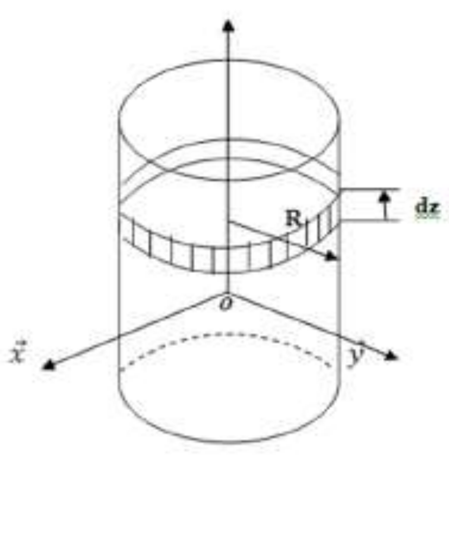
$$= \frac{1}{4}\rho\pi[R_2^4.H - R_1^4.H] + \frac{\rho\pi}{12}[R_2^2.H^3 - R_1^2H^3]$$

$$= \frac{1}{4}\frac{M}{H(R_2^2 - R_1^2)}[R_2^4H - R_1^4H] + \frac{1}{12}\frac{M}{H(R_2^2 - R_1^2)}H^3[R_2^2 - R_1^2]$$

$$= \frac{1}{4}M\frac{R_2^4 - R_1^4}{R_2^2 - R_1^2} + \frac{1}{12}MH^2$$

$$\boxed{A = B = \frac{M}{4}(R_2^2 + R_1^2) + \frac{M}{12}H^2}$$

b) Cylindre mince d'épaisseur faible



$(\vec{x}, \vec{y}, \vec{z})$ est une base principale d'inertie

$\Longrightarrow$ la matrice d'inertie est diagonale

$$[I_{(S)}]_{(o,\vec{x},\vec{y},\vec{z})} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

$\vec{x}$ $\vec{y}$

$(o, \vec{x})$ et $(o, \vec{y})$ jouent le même rôle $\Longrightarrow A = B$

On a : $ds = 2\pi.R.dz$

$$C = \int (x^2 + y^2)dm \qquad ; \quad x^2 + y^2 = R^2$$

$\int R^2.dm = R^2 \int dm$

On a : $dm = \sigma.ds = \sigma.2\pi.R.dz$

$$C = 2\pi R^2 . \sigma . R \int dz = 2\pi R^3 \sigma \int_{-H/2}^{H/2} dz$$

$$= 2\pi . R^3 . H . \sigma$$

$$\rho = \frac{M}{S} = \frac{M}{2\pi R.H} \Longrightarrow C = 2\pi R^3 . H . \frac{M}{2\pi .R.H} = MR^2$$

$$\boxed{C = MR^2}$$

$$\boxed{A = B = \frac{MR^2}{2} + \frac{MH^2}{12}}$$

2ème méthode:

Pour un cylindre creux de rayon *(R_1 et R_2)*

$$C = \frac{1}{2} M \frac{R_2^4 - R_1^4}{R_2^2 - R_1^2} \quad avec \;\; R_2 = R_1 + e$$

$$= \frac{1}{2} M \frac{(R_1 + e)^4 - R_1^4}{(R_1 + e)^2 - R_1^2}$$

$$= \frac{1}{2} \quad \frac{R_1^4 \left[1 + \frac{e}{R_1}\right]^2 - R_1^4}{R_1^2 \left[1 + \frac{e}{R_1} - R_1^2\right]} = \frac{1}{2} \; M \; \frac{R_1^4}{R_1^2} \; \frac{\left[(1 + \frac{e}{R_1})^4 - 1\right]}{\left[(1 + \frac{e}{R_1})^2 - 1\right]}$$

Développement limité

$\Longrightarrow$

$$C = \frac{M}{2} R_1^2 \; \frac{1 + 4 . \frac{e}{R_1} - 1}{1 + 2 . \frac{e}{R_1} - 1}$$

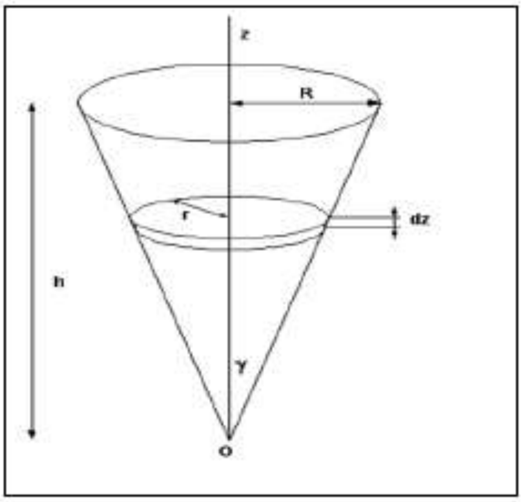
$$C = \frac{M}{2} \; . \; R_1^2 \; \frac{4 . \frac{e}{R_1}}{2 . \frac{e}{R_1}}$$

$$C = MR_1^2 \; , \; R = R_1 \;\; \Longrightarrow \;\; C = MR^2$$

Et

$$A = B = \frac{MR^2}{2} + \frac{MH^2}{12}$$

c) Cône creux de rayon R et de hauteur H



$(o,\vec{x},\vec{z})$ et $(o,\vec{y},\vec{z})$ deux plans de symétrie

$\Rightarrow$. $(o,\vec{y})$ et $(o,\vec{x})$ sont deux axes principaux d'inertie.

$\vec{x}\wedge\vec{y}=\vec{z} \Rightarrow (o,\vec{z})$ *est un axe pricipal d'inertie*

$$\Rightarrow [I_{(S)}]_{(o,\vec{x},\vec{y},\vec{z})}=\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

$(o,\vec{x})et(o,\vec{y})$ *jouent le même rôle* $A=B$

$$C=\int (x^2+y^2)dm=\int r^2.dm \quad on\ a\ \tan\alpha=\frac{R}{H}=\frac{r}{z}\Rightarrow r=\frac{R}{H}.z$$

$$dm=\sigma.ds=\sigma.2\pi r\ .dz$$

$$C=\int r^2.\sigma.ds=\sigma.\int r^2.2\pi r.dz$$

$$C=\ 2\pi.\sigma\int r^3.dz$$

$$=2\pi.\sigma.\int\frac{R^3}{H^3}.z^3.dz$$

$$C=\int r^2.\sigma.ds\ \ =2\pi\sigma.\frac{R^3}{H^3}.\int_0^H z^3dz$$

$$=2\pi\sigma.\frac{R^3}{H^3}.\frac{H^4}{4}$$

$$=\pi.\sigma.R^3.\frac{H^4}{2}\quad;\quad \sigma=\frac{M}{S}=\frac{M}{\pi RH}$$

$$=\pi\frac{M}{\pi RH}R^3=\frac{MR^2}{2}$$

$$\boxed{C=\frac{MR^2}{2}}$$

$$A=B=\frac{1}{2}C\int z^2dm$$

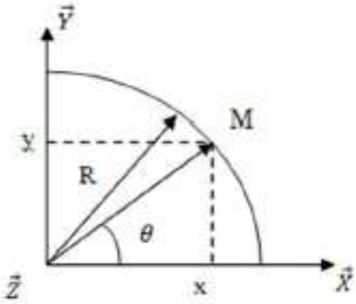
$$\int z^2 dm = \sigma.\int z^2.ds = \sigma.\int z^2.2\pi\, r dz$$

$$= 2\pi\left(\frac{R}{H}\right).\sigma\int_0^H z^3.dz = 2\pi\left(\frac{R}{H}\right).\sigma.\frac{H^4}{4}$$

$$= 2\pi\left(\frac{R}{H}\right).\frac{M}{\pi RH}.\frac{H^4}{2} \quad = \quad \frac{MH^2}{2}$$

$$\boxed{A = B = \frac{1}{4}MR^2 + \frac{1}{2}MH^2}$$

d) Quart de cercle de rayon R



On a z = 0

$A = \int y^2.dm,\ B = \int x^2 dm,\quad C = \int (x^2 + y^2)\,dm$

$$[I_{(S)}]_o = \begin{bmatrix} A & -F & 0 \\ -F & B & 0 \\ 0 & 0 & C \end{bmatrix} \qquad F = \int x\,y\ \ dm$$

Ona

$$C = \int R^2.dm \qquad avec \qquad dm = \lambda.dl \quad = \lambda R\ d\theta$$

$$C = \int R^2\lambda R.\ d\,\theta = \lambda R^3\int_0^{\pi/2} d\theta = \lambda R^3\frac{\pi}{2}$$

$$\lambda = \frac{M}{L} = \frac{M}{\frac{\pi}{2}R}\frac{2M}{\pi R}$$

$$C = 2\frac{M}{\pi R}.R^3\frac{\pi}{2} \qquad\qquad C = \ MR^2$$

$$A + B = \int (x^2 + y^2) dm = C \qquad\qquad or \quad A = B$$

$\Longrightarrow$

$$\boxed{A = B = \frac{C}{2} = \frac{MR^2}{2}}$$

$$F = \int xy\ dm = \int R\cos\theta.R\sin\theta\ dm$$

$$= R^2\int \cos\theta\sin\theta\,.dm$$

$$= \lambda R.R^2\int_0^{\pi/2}\cos\theta\ \sin\theta\ \ d\theta$$

$$= \lambda . R^3 \int_0^{\pi/2} \cos\theta \ \sin\theta \ d\theta$$

$$= \lambda R^3 \int_0^{\pi/2} \frac{\sin 2\theta}{2}$$

$$= \lambda R^3 \ \frac{1}{2}\left[-\frac{1}{2} \ \cos 2\theta\right]_0^{\pi/2}$$

$$= \lambda \frac{R^3}{4}[-\cos 2\theta]_0^{\pi/2}$$

$$= \lambda \frac{R^3}{4}[-\cos\pi + \cos 0]$$

$$= \lambda \frac{R^3}{2} \qquad ; \qquad \lambda = 2\frac{M}{\pi . R}$$

$$F = 2\frac{M}{\pi . R} \ \frac{R^3}{2} = M\frac{R^2}{\pi}$$

$$\boxed{F = \frac{MR^2}{\pi}}$$

*

Exercice 2

1) $(o, \vec{y}, \vec{z})$ et $(o, \vec{y}, \vec{z})$ sont deux plans de symétrie matérielle

$\Longrightarrow (o, \vec{z})$ et $(o, \vec{y})$ sont deux axes principaux d'inertie

Le troisième axe : $\vec{x} \wedge \vec{y} = \vec{z}$

$\Longrightarrow$ $(o, \vec{z})$ est le $3^{\text{ème}}$ axe principal

$\Longrightarrow$ la base $(o, \vec{y}, \vec{z})$ est une base principale d'inertie

2) ***Position du centre d'inertie***

Systéme directe :

$$\overrightarrow{OG} = \frac{1}{M}(m_1\overrightarrow{OG_1} + m_2\overrightarrow{OG_2}$$

$$M = m_1 + m_2 = \rho(v_1 + v_2)$$

$$m_1 \rho v_1 \quad et \quad m_2 = \rho v_2$$

$$\overrightarrow{OG} = \frac{1}{v_1+v_2}(v_1\overrightarrow{OG_1} + v_2\overrightarrow{OG_2}$$

$$v_1 + v_2 = \pi R^2 \frac{2}{3}\pi R^3$$

$$\boxed{v_1 + v_2 = \pi R^2 H \ \frac{2}{3}\pi R^3}$$

$$v_1 = \pi R^3 . H \ , \ \overrightarrow{OG_1} = \frac{H}{2}\vec{z}$$

$$v_2 = \frac{2}{3}\pi R^3 \ , \qquad cherchons \ : \ \overrightarrow{OG_2}$$

$(o, \vec{y}, \vec{z})$ et $(o, \vec{y}, \vec{z})$ sont deux plans de symétrie matérielle $\Longrightarrow$ $G \in (o, \vec{z})$

$$\Longrightarrow z_G = \frac{1}{m^2}\int z \, dm$$

$$= \frac{1}{v^2} \int z\, dv$$

$$dv = \pi r^2 dz$$

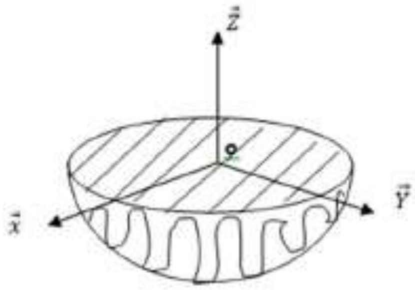
$$on\ a\ \ \tan\theta \frac{r}{z} \Longrightarrow \mathrm{r} = \mathrm{z}.\tan\theta$$

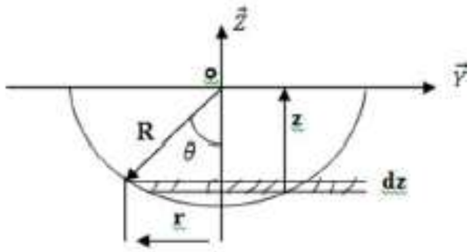
$$dv = \pi \tan^2\theta \,.\, z^2 . dz$$

$$on\ a\ \ \cos\theta = \frac{z}{R}\ ; \quad \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1-\cos^2\theta}{\cos^2\theta}$$

$$\tan^2\theta = \frac{1-\frac{z^2}{R^2}}{\frac{z^2}{R^2}} = \frac{R^2-z^2}{z^2}$$

$$\boxed{dv = \pi(R^2 - z^2)dz}$$





$$z_G = \frac{1}{\frac{2}{3}\pi R^3} \pi \int (R^2 - R^2) z\, dz$$

$$= \frac{3}{2\mathrm{R}^3} \int_0^{\mathrm{R}} (\mathrm{R}^2 \mathrm{z} - \mathrm{z}^3) \mathrm{dz}$$

$$= \frac{3}{2\mathrm{R}^3} \left[\mathrm{R}^2 . \frac{\mathrm{z}^2}{2} - \frac{\mathrm{z}^4}{4} \right]_0^{\mathrm{R}}$$

$$z_G = \frac{3}{2\mathrm{R}^3} \left[\frac{2\mathrm{R}^4}{4} - \frac{\mathrm{R}^4}{4} \right]$$

$$= \frac{3}{2\mathrm{R}^3} . \frac{\mathrm{R}^4}{4} = \frac{3}{8}\mathrm{R} \qquad \mathrm{z_G} = \frac{3}{8} R$$

$$\boxed{\overrightarrow{\mathrm{OG}_2} = -\frac{3}{8} R\vec{\mathrm{z}}}$$

$$\overrightarrow{OG} = \frac{1}{\pi R^2 . H + \frac{2}{3}\pi R^3} \left[\pi R^2 . H . \frac{H}{2} \vec{z} - \frac{2}{3} \pi R^3 . \frac{3}{8} R \right] \vec{z}$$

$$= \frac{1}{\pi R^2 H + \frac{2}{3}\pi R^3} \left[\pi R^2 . \frac{H^2}{2} - \frac{1}{4} \pi R^4 \right] \vec{z}$$

$$= \frac{3}{3R^2H + 2R^3} \cdot \left(\frac{2R^2H^2 - R^4}{4}\right)\vec{z}$$

$$\boxed{OG = \frac{3(2R^2H^2 - R^4)}{4(3R^2H + 2R^3)}\vec{z}} \qquad \boxed{z_G = \frac{3(2R^2H^2 - R^4)}{4(3R^2H + 2R^3)}}$$

3) Matrice d'inertie de (S) en o

$$[I_{(S)}]_O = [I_1]_O + [I_2]_O = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}_{(o,\vec{x},\vec{y},\vec{z})}$$

$$A = A_1 + A_2 \qquad A_1 = \frac{1}{2}\frac{m_1R^2}{4} + \frac{m_1H^2}{3}$$

$$B = B_1 + B_2 \qquad B_1 = \frac{m_1R^2}{4} + \frac{m_1H^2}{3}$$

$$C = C_1 + C_2 \qquad C_1 = \frac{m_1R^2}{2}$$

Calculons $[I_2]_O$: matrice d'inertie de la sphère

$$on\ a \quad A_2 = \int (y^2 + z^2)dm \ ; \ B_2 = \int (x^2 + z^2)\,dm \ ; \ C_2 \int (x^2 + y^2)dm$$

$or\ (o,\vec{x}), (o,\vec{y})$ et $(o,\vec{y})$ jouent le même rôle

$$\Longrightarrow \ A_2 = B_2 = C_2$$

$$A_2 + B_2 + C_2 = 2.\int (x^2 + y^2 + z^2)dm$$

$$\Longrightarrow \ A_2 + B_2 + C_2 = 2I_o$$

$$\boxed{\Longrightarrow A_2 = B_2 = C_2 = \frac{2}{3}I_o}$$

$$on\ a: \quad I_o = \int (x^2 + y^2 + z^2)dm = \int r^2dm$$

$$dm = \rho dv$$

$$dm = \rho r^2dr.\sin\theta\ d\theta.\ d\varphi$$

$$I_o = \rho\int_0^R r^4dr\int_0^\pi \sin\theta\,d\theta\int_0^\pi d\varphi$$

$$I_o = \rho \frac{R^5}{5} 2\pi$$

$$I_o = \frac{2}{5}\pi\rho R^5 \qquad \text{or} \qquad \rho = \frac{m_2}{v} = \frac{m_2}{\frac{2}{3}\pi R^3} = \frac{3m_2}{2\pi R^3}$$

$$I_o = \frac{2}{5}\pi \frac{3m_2}{2\pi R^3} R^5$$

$$\boxed{I_o = \frac{3}{5} m_2 R^2} \Rightarrow \boxed{A_2 = B_2 = C_2 = \frac{2}{5} m_2 R^2}$$

$$\Rightarrow A = \frac{m_1 R^2}{4} + \frac{m_1 H^2}{3} + \frac{2}{5} m_2 R^2$$

$$B = \frac{m_1 R^2}{4} + \frac{m_1 H^2}{3} + \frac{2}{5} m_2 R^2$$

$$C = \frac{m_1 R^2}{2} + \frac{2}{5} m_2 R^2$$

$$on\ a\ \ M = m_1 + m_2 = \rho v = \rho(\pi R^2 H + \frac{2}{3}\pi R^3)$$

$$= \rho\pi(R^2 H + \frac{2}{3} R^3)$$

$$A = \rho\pi R^2 H \frac{R^2}{4} + \rho\pi R^2 H.\frac{H^2}{3} + \frac{2}{5}.\rho\frac{2}{3}\pi R^3 R^2$$

$$A = \rho\pi\left[\frac{R^4 H}{4} + \frac{R^2 H^3}{3} + \frac{4}{15} R^5\right]$$

$$A = \frac{M}{R^2 H + \frac{2}{3} R^3}\left[\frac{R^4 H}{4} + \frac{R^2 H^3}{3} + \frac{4R^5}{15}\right]$$

$$A = \frac{3M}{3R^2.H + 2R^3} \ .\ R^2(\frac{R^2 H}{4} + \frac{H^3}{3} + \frac{4}{15} R^3)$$

$$\boxed{A = \frac{3M}{3H + 2R}(\frac{R^2 H}{4} + \frac{H^3}{3} + \frac{4}{15} R^3)}$$

<u>A=B</u>

$$C = \rho\pi R^2 H \frac{R^2}{2} + \frac{2}{5}\rho\frac{2}{3}\pi R^3.R^2$$

$$C = \rho\pi R^2(\frac{HR^2}{2} + \frac{4}{15} R^3)$$

$$C = \frac{M}{R^2H + \frac{2}{3}R^3} R^2 (\frac{HR^2}{2} + \frac{4}{15}R^3)$$

$$C = \frac{3M}{3H + 2R} (\frac{HR^2}{2} + \frac{4R^3}{15})$$

4) *Calculons* $[I_{(S)}]_G = \begin{bmatrix} A_G & 0 & 0 \\ 0 & B_G & 0 \\ 0 & 0 & C_G \end{bmatrix}_{(G,\vec{x},\vec{y},\vec{z})}$

$$z_G = \frac{3}{4} (\frac{2R^2H^2 - R^4}{3R^2H + 2R^3})$$

$$A_G = A - M_G^2$$

$$B_G = B - M_G^2$$

$$C_G = C$$